Minkowski Spacetime

by Ronald Koster

(http://www.ronaldkoster.net)

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1 Introduction

This document describes the reformulation of the Lorantz Transformation as given by Minkowski. This new formulation gave rise to the concept of *spacetime* as we know today.

2 Rotation

First we consider the situation when a co-ordinate systems becomes rotated with respect to another co-ordinate system.

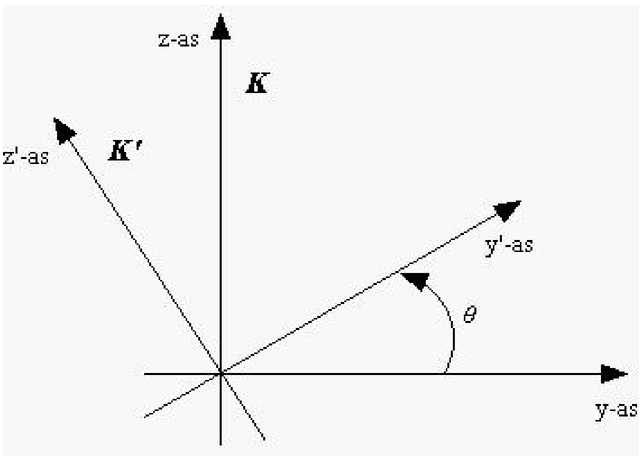

Figure 1: Rotated co-ordinate system

As can be seen in figure (1) the K co-ordinate system has rotated with respect to the K' system along the x-axis. The following relationship can easily be deduced:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

This is called a co-ordinate transformation.

3 Lorentz Transformation

Given two co-ordinate systems as shown in figure (2).

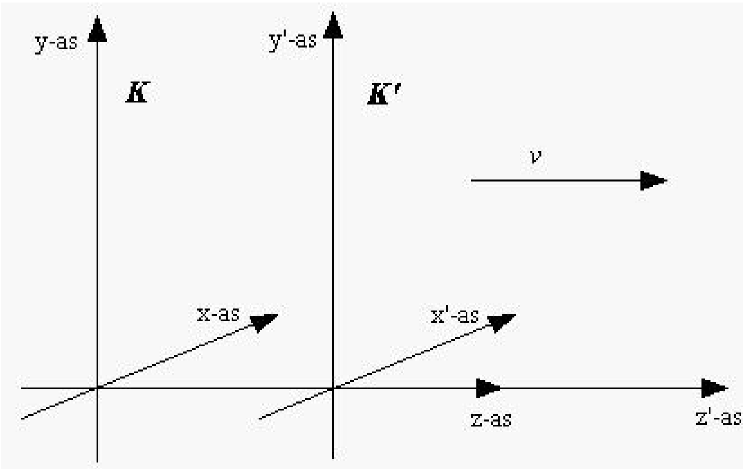

Figure 2: Moving co-ordinate systems

The K' system moves with velocity v along the z-axis. According to Einstein the co-ordinate tranformation between the two systems is given by the Lorentz Transformation (LT):

$$x' = x \tag{2}$$
$$y' = y \tag{3}$$
$$z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4}$$
$$t' = \frac{t - \frac{zt}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

4 Minkowski's formulation

For now lets focus on the z and t co-ordinates. Then the following relationship occurs:

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} 1/\gamma & -\beta c/\gamma \\ -\beta/\gamma c & 1/\gamma \end{pmatrix} \begin{pmatrix} z' \\ t' \end{pmatrix} \tag{6}$$

using

$$\beta = \frac{v}{c} \tag{7}$$
$$\gamma = \sqrt{1 - \frac{v^2}{c^2}} \tag{8}$$

Substituting the relations

$$
\begin{aligned}
x_3 &= z && (9) \\
x_4 &= ict && (10)
\end{aligned}
$$

gives

$$
\begin{pmatrix} x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 1/\gamma & i\beta/\gamma \\ -i\beta/\gamma & 1/\gamma \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{11}
$$

Now introduce

$$
\cosh\theta = \frac{1}{\gamma} \tag{12}
$$

Using the relation $\cosh^2 x - \sinh^2 x = 1$, it can be decduced that

$$
\begin{aligned}
\sinh\theta &= \frac{\beta}{\gamma} && (13) \\
\tanh\theta &= \beta && (14)
\end{aligned}
$$

Finally, substituting this results in

$$
\begin{aligned}
\begin{pmatrix} x_3' \\ x_4' \end{pmatrix} &= \begin{pmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} && (15) \\
&= \begin{pmatrix} \cos i\theta & \sin i\theta \\ -\sin i\theta & \cos i\theta \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} && (16)
\end{aligned}
$$

Using the relations $\cos i\theta = \cosh\theta$ and $\sin i\theta = i\sinh\theta$.

The total transformation matrix thus is

$$
\begin{aligned}
\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos i\theta & \sin i\theta \\ 0 & 0 & -\sin i\theta & \cos i\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} && (17) \\
&= M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} && (18)
\end{aligned}
$$

Now compare this with the rotation formula (1) and notice the similarity. The Lorentz Transformation appears to be merely a rotation in the complex co-ordinate system (x_1, x_2, x_3, x_4) along the complex angle $i\theta$. A remarkable observation! Formulated in this way the time co-ordinate x_4 acts as just another space co-ordinate. This initiated the idea that space and time are even more intermixed than the Lorentz Transformation in its original formulation already suggested. The complex co-ordinate system (x_1, x_2, x_3, x_4) is what is called *Minkowski spacetime*, or simply *spacetime*.